Intermediate-range coupling generates low-dimensional attractors deeply in the chaotic region of one-dimensional lattices

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Abstract

Properties of intermediate-range coupling are studied in one-dimensional coupled map lattices (CMLs). Phase diagrams have been constructed which describe the relationship between the range of coupling and coupling strength. A delicate low-dimensional attractor is emerging for non-global interactions in the case of weak coupling, while the leading Lyapunov exponent is a large positive number. © 1998 Elsevier Science B.V.

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1. Introduction

Spatio-temporal dynamics of complex, nonlinear systems has been studied intensively during the recent years, including fluid flows, crystal growth, coupled optical systems, evolutionary information processing, neuro-dynamics, etc. In modeling such systems, CMLs utilise continuous state-space and discrete time and space coordinates [1–4]. Lattice elements in a CML are obtained by coarse-graining the original microscopic quantities. Coarse-grain models represent a connection between the microscopic world and macroscopic observations and correspond to the level of our knowledge about the physical processes. Coupled map lattice theory grew out of studies on collective movements of coupled oscillators [1]. CMLs with local, direct-neighbour coupling can be regarded as approximations to the diffusion process. Globally coupled maps are related to mean field interactions and they are defined as [3]

\[ x_{n+1}(i) = (1 - \varepsilon) f(x_n(i)) + \frac{\varepsilon}{N} \sum_{k=1}^{N} f(x_n(k)), \quad (1) \]

where \( x_n(i) \) is the field value at location \( i \) and time step \( n \), \( i = 1, \ldots, N \); \( N \) is the size of the one-dimensional lattice.

Globally coupled maps can be written in the form

\[ x_{n+1}(i) = (1 - \varepsilon) f(x_n(i)) + \varepsilon H_n, \quad (2) \]

where \( H_n \) denotes the mean field

\[ H_n = \frac{1}{N} \sum_{k=1}^{N} f(x_n(k)). \quad (3) \]

The coupling strength \( \varepsilon \) varies from 0 to 1. \( f(x) \) is a properly selected nonlinear mapping function. In the
following discussions we use the well-known logistic function with $0 \leq a \leq 2$,

$$f(x) = 1 - ax^2. \tag{4}$$

In locally and globally coupled lattices various types of temporal behaviours have been described, ranging from fixed points and limit cycles to collective quasi-periodic and chaotic oscillations [3–8]. The presence of low-dimensional collective behaviours in spatially extended systems is a controversial issue. Contrary to earlier theoretical studies predicting the absence of global collective behaviours in the case of semi-localised coupling, quasi-periodic collective dynamics has been observed in such systems [6,9–10]. Further studies showed that the onset of macroscopic collective behaviour can be attributed to the window structure of the bifurcation maps [11,12].

In this Letter the importance of intermediate-range effects is emphasised. Intermediate-range coupling can be described by a modified version of Eq. (3), in which the mean field coupling $H_n$ is replaced by the term

$$h^n_K(i) = \frac{1}{2K+1} \sum_{k=-K}^{K} f(x_n(i+k)), \tag{5}$$

where $K$ is the coupling range, $1 < K < N/2$.

Intermediate coupling is closely related to the topology of the physical space and it plays a crucial role in solving practical problems. Until now, however, relatively little attention is paid to intermediate-range effect due to the complexity of the required analysis. Pioneering works in this field include studies on spatial correlations based on coupled Ginzburg–Landau-type oscillators and investigations on the effect of higher spatial dimensionality on the collective behaviours [6–8,13–14].

In the field of neuro-dynamics it has been realized only very recently that network architectures with non-local connectivity and adaptive structure are key elements of artificially intelligent neural networks [15,16]. In addition to the numerical data-processing features, structured neural networks create the key conditions for the emergence of intelligence, which is manifested in the form of high-level symbolic knowledge, causal reasoning and symbolic rules. Results obtained with studying intermediate-range coupling in CMLs are directly related to the emergence of intelligent behaviour in neural networks.

2. Experiments with intermediate-range coupling

2.1. Background

The work introduced in the present Letter is closely related to the research by Sinha et al. who analysed the effect of increasing extent of coupling using a one-dimensional lattice of logistic maps. When the number of coupled chaotic elements increases, a collective behaviour appears in the form of well-defined peaks in the power spectra of the mean field oscillations. The mean-square deviation $\sigma$ of the mean field has been evaluated as follows,

$$\sigma = \frac{1}{L} \sum_{n=1}^{L} (x_n - \bar{x})^2, \tag{6}$$

where $x_n$ is the mean field value at time $n$; $L$ is the number of time steps used in the temporal averaging. In Ref. [10], a linear relationship has been found between $\sigma$ and the number of coupled elements over a wide range of $K$ values. It is briefly stated in footnote 7 in Ref. [10] that, surprisingly, the peak of the power spectrum is sharper for some intermediate coupling levels than for the globally coupled map. The observation that the peaks of the power spectra are of maximum sharpness for a non-global coupling level is remarkable, because it might indicate the presence of a pronounced intermediate-range behaviour.

2.2. Statistical analysis of mean field oscillations

The goal of this section is to introduce the results obtained on the relationship between the extent of coupling $K$ and the statistical features of the mean field fluctuations. These results link our studies with previous works on intermediate-range coupling [10,12].

We have conducted a systematic study of coupled maps for a wide range of parameters: system size $N$, extent of coupling $K$, coupling strength $\epsilon$, and control parameter $a$. In the studies, a periodic boundary condition has been used. Experiments with a fixed system size $N = 1001$ and a given control parameter value $a = 1.99$ are described in this Letter. In the experi-
ments, $K$ and $\varepsilon$ have been varied from 0 to $N$ and from 0 to 1, respectively.

The power spectra of the mean field develop prominent peaks as $K$ increases from local to global coupling. The change of the magnitude of the oscillations is characterised by the $\sigma$ value of the mean field. In Fig. 1a, $\sigma$ is depicted as the function of $K$ for two $\varepsilon$ values. In the case of $\varepsilon = 0.1$, the $K$ versus $\sigma$ curve is approximately linear in accordance with previous observations [10]. For $\varepsilon = 0.14$, however, a more complicated curve is observed. Namely, $\sigma$ increases slowly at first, followed by a quick growth at $K$ values between 100 and 200. Finally, $\sigma$ increases only slightly when $K$ is large. The peculiar behaviour of the mean field at intermediate coupling $K$ is even more prominent in Fig. 1b, where the peak value of the power spectra $S_{\text{max}}$ is shown along a logarithmic axis. $S_{\text{max}}$ saturates for intermediate and large $K$ values (as in curve A in Fig. 1b), or it can have a broad maximum over intermediate $K$ values (in curve B in Fig. 1b).

Results shown in Figs. 1a,b indicate that the effects caused by intermediate coupling are more complicated than originally thought. The nature of intermediate-range coupling is scrutinised further in this Letter by building phase diagrams in the $K$ versus $\varepsilon$ space.

2.3. General features of phase diagrams

The chaotic behaviour of the lattice dynamics is analysed through the leading Lyapunov exponent $\lambda_1$. We determine $\lambda_1$ from the governing equation by a standard perturbation method [17]. In the calculations, $\varepsilon$ varies in the range between 0 and 1 and $K$ takes values from local to global coupling. The results are summarised in Fig. 2 for a lattice of size $N = 1001$. The resolution of Fig. 2 is $\Delta \varepsilon = 0.1$ and $\Delta K = 10$. Lighter tones mark positive Lyapunov exponents, while dark-grey and black colours denote negative values, as indicated by the grey-scale bar on the right of Fig. 2. Experiments have been performed with iteration numbers from several thousands to $10^5$. Data in Fig. 2 have been calculated using 1000 iterations after omitting 1000 initial transients. The results of a more refined analysis of the distribution of the leading Lyapunov exponents are given in Fig. 3 for a model with $N = 101$. The resolution of the calculations is $\Delta \varepsilon = 0.02$ and $\Delta K = 1$ in Fig. 3. The other parameters are the same as in Fig. 2. Figs. 2 and 3 are remarkably similar when considering the relative coupling range.
coordinate $K/N$ instead of $K$.

Fig. 2 represents a phase diagram in the $K$ versus $\varepsilon$ space and it can be divided into several characteristic sections. There is a region with strictly positive $\lambda_1$ for large $\varepsilon$ and large $K$ values, which describes coherent states. This round-shaped coherent region is surrounded by a "ring" segment of negative or very small positive $\lambda_1$'s corresponding to fixed points, limit cycles and quasi-periodic behaviours. In addition to this ring, $\lambda_1$ diminishes also in a straight and narrow basin spanning from $\varepsilon \approx 0.2$ to $0.4$ and running through almost the whole range of $K$. The behaviour inside this basin is similar to the behaviour in the ring segment described above. There is a secondary ring segment with diminishing Lyapunov exponents inside the turbulent region of medium-to-large $\varepsilon$ and intermediate $K$ values. Traces of a 3rd ring segment are also seen next to the second ring at somewhat lower $K$ values. The rest of the parameter combinations correspond to ordered, partially ordered, intermittent or turbulent regimes. Turbulent regime is the most pronounced at small $\varepsilon$ or small $K$ values with de-synchronised states of the individual lattice elements.

Owing to computational constraints, we do not attempt to describe the details of the phase diagram in the present study. This should be the subject of future extensive research. In the last part of this Letter, we elaborate on two particular points, which can help to understand better the structure of the phase diagrams. At first, the coherent attractor states will be analysed, followed by the description of a peculiar collective motion found deeply in the turbulent region.

Fig. 3. Distribution of the leading Lyapunov exponent $\lambda_1$ in the $K$ versus $\varepsilon$ space; $N = 101$ and $\alpha = 1.99$. Notations are the same as in Fig. 2.

2.4. Coherent states: the case of strong coupling

In this section we concentrate on the investigation of the coherent attractor states. The stability of the coherent attractors is analysed using the Jacobi matrices $J_n$. For the globally coupled lattice, the Jacobi matrices have the form

$$J_n = f'(x_n) \left( (1 - \varepsilon) I + \frac{\varepsilon}{N} D \right),$$

(7)

where $I$ is an $N \times N$ unit diagonal matrix, and all the elements of matrix $D$ are 1. The stability condition of the coherent attractors in the globally coupled lattice is given by [1]

$$\lambda_0 + \log(1 - \varepsilon) < 0,$$

(8)

where $\lambda_0$ is the Lyapunov exponent of a single map. For the logistic function with $\alpha = 1.99$ and $\lambda_0 = 0.6892$, the stability condition is $\varepsilon > \varepsilon_* = 0.4980$. Here $\varepsilon_*$ is the stability threshold of the coherent attractor.

In the case of intermediate coupling ($1 < K < N/2$), the Jacobi matrices write

$$J_n^{(k)} = f'(x_n) \left( (1 - \varepsilon) I + \frac{\varepsilon}{(2K + 1)} D_k \right),$$

(9)

where $D_k$ is a sparse $N \times N$ connectivity matrix of elements 0 and 1. The structure of the matrix $D_k$ is schematically depicted in Fig. 4. Eq. (9) is a generalisation of Eq. (7). In the case of $N = 2K + 1$, Eq. (9) yields the globally coupled result given by Eq. (7).

The stability condition of the globally coupled map, given by Eq. (8), cannot be easily generalized to intermediate-range coupling. The main reason of
Fig. 5. Schematic distribution diagram of the coherent phase. Dashed: edge of the coherent stability obtained by the simplified condition; black color denotes region calculated by numerical simulation.

the problems is the more complicated structure of the eigenvectors and eigenvalues of the Jacobians with intermediate-range coupling. For globally coupled maps, there is a leading eigenvalue, while the other \( N - 1 \) eigenvalues are identical and represent an \((N - 1)\)-fold degeneracy [1]. The eigenvector belonging to the first eigenvalue is uniform; therefore, an amplification along this eigenvector cannot destroy the coherence [1]. In the case of intermediate coupling, we still have a leading eigenvalue and a corresponding uniform eigenvector \([1, 1, \ldots, 1]^T / \sqrt{N}\). This can be shown using quite straightforward algebra. The remaining \( N - 1 \) eigenvalues, however, will not be identical, i.e., there is no \((N - 1)\)-fold degeneracy in CMLs with intermediate-range coupling. Alternative ways of system analysis have to be explored in this case.

We use two methods to describe the coherent states in the case of intermediate-range coupling:

(i) performing direct numerical simulations to determine the boundaries of the coherent attractor states; (ii) introducing some approximations which could yield at least a qualitative understanding of the rounded shape of the coherent region in the phase diagram.

Results of numerical simulations for a lattice with \(N = 101\) are shown in Fig. 5. The black area in Fig. 5 denotes the extent of stability region which agrees well with the region of the coherent attractor (\(\lambda_1 > 0\)) located in the upper-right corner of Figs. 2 and 3.

A rough estimation of the stability margin can be obtained from the second term of Eq. (9). In the case of global coupling, the stability condition is \(\varepsilon > \varepsilon_*\). Here \(\varepsilon_*\) is the stability threshold of the coherent attractor of logistic maps as is described previously in connection with Eq. (8). By decreasing \(K\), the factor \(\varepsilon/(2K + 1)\) starts to increase. This increase can be incorporated into an “effective coupling strength” \(\varepsilon_{\text{eff}} = \varepsilon N/(2K + 1)\). By neglecting changes in matrix \(D_k\) and neglecting the influence of the first term in Eq. (9), we can approximate the stability condition as \(\varepsilon_{\text{eff}} > \varepsilon_*\). The corresponding stability region is located to the right of the dashed line in Fig. 5, which shows a good qualitative agreement with the stability region obtained by direct numerical simulations. The discrepancy is marked in gray in Fig. 5. This discrepancy is due to the introduced approximations, i.e., neglecting the change in the structure of matrix \(D_k\) and also neglecting the contribution of the first term in Eq. (9).

2.5. Non-trivial collective motions in turbulent regime: the case of weak coupling

We turn our attention to weak coupling, when turbulent behaviour dominates the dynamics. Basically, this is the region with \(\varepsilon \leq 0.2\) in Figs. 2 and 3. Here the leading Lyapunov exponent is strictly positive (it is larger than 0.2), therefore, chaotic regime prevails. In Figs. 6a–c, the first return plots of the mean field are shown for \(K\) values 10, 170, and 500, respectively. The parameters of the calculations are \(N = 1001\), \(\varepsilon = 0.14\), and \(\alpha = 1.99\). At intermediate coupling level (\(K = 170\)), a non-trivial collective motion is observed in the form of a torus-like structure; see Fig. 6b. This torus structure is present over the range \(100 \leq K \leq 450\). In the case of (semi-) local and global coupling depicted in Figs. 6a,c, the torus is not seen. This observation indicates that the observed torus is indeed an intermediate-range effect. By increasing \(N\), the peculiar loop becomes more clear and its position remains unchanged. This conclusion is illustrated in Fig. 7, where the torus-shaped collective behaviour is shown for \(N = 20000\) and \(K = 3400\).
3. Conclusions

In summary, it can be concluded that the investigation of intermediate-range coupling effects opens an exciting new direction of CML studies. The results introduced in this Letter indicate some ways of studying collective behaviours of spatio-temporal chaos in one spatial dimension.

(i) In the case of strong coupling ($\varepsilon > 0.5$), the maximum Lyapunov exponent approaches zero and becomes negative at certain intermediate $K$ ranges. This observation indicates that quasi-periodic behaviour, limit cycle oscillations and fixed points can occur in this parameter range.

(ii) For a weak coupling ($\varepsilon < 0.2$), on the other hand, the Lyapunov exponent remains always positive. Nevertheless, low-dimensional collective behaviours are observed at these parameters as well. The relatively large Lyapunov exponents ($> 0.2$) show that this low-dimensional collective behaviour is essentially different from the non-trivial quasi-periodic oscillations found previously [2].

(iii) Similar phenomena have been described at higher space dimensions [6–8]. Our results are obtained in one-dimensional lattices, which is very advantageous from a computational viewpoint. Owing to the lower computational burden in such systems, it is feasible to conduct a systematic exploration of the space of parameters: system size, coupling size and coupling extent, etc. The present work is just an initial step in that direction.

Intermediate-range effects represent a delicate balance between synchronization through global coupling and localised fragmentation due to individual chaotic dynamics. The interaction of these two extremes can yield peculiar intermediate-range effects, which play a key role in self-organisation and in the emergence of structures in complex systems. Continued research in this field can reveal further exciting new features of spatio-temporal chaos.
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